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On quantum revivals and quantum fidelity. A semiclassical approach

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Abstract
The aim of this paper is three-fold: first, to establish in a clear and rigorous way a formula proposed heuristically by Mehlig and Wilkinson for the metaplectic operators corresponding to a given symplectic transformation in classical phase space. Second, this formula is applied to the study of quantum recurrences, which has attracted a great deal of interest in recent years (see [35] for a complete account of the recent approaches). The return probability is given by the squared modulus of the overlap between a given initial wavepacket and the corresponding evolved one; quantum recurrences in time can be observed if this overlap is unity. We provide some conditions under which this is semiclassically achieved taking as the initial wavepacket a coherent state localized on a closed orbit of the corresponding classical motion. Third, we start a rigorous approach of the 'quantum fidelity' (or the Loschmidt echo): it is the squared modulus of the overlap of an evolved quantum state with the same state evolved by a slightly perturbed Hamiltonian. It has attracted a great deal of interest in the last decade, for the purpose of 'quantum chaos' problems, and in quantum computation analysis. However, the results are most of the time not entirely conclusive, and sometimes even contradictory. Thus it is useful to start a rigorous approach to this problem. The decrease in time of the quantum fidelity measures the sensitivity of quantum evolution with respect to small perturbations. Starting with suitable initial quantum states, we develop a semiclassical estimate of this quantum fidelity in the linear response framework (appropriate for the small perturbation regime), assuming some ergodicity conditions on the corresponding classical motion.

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1. Introduction

The revival of quantum wavepackets along the time evolution has attracted much recent interest (see [35] and references therein), and we want to stress here how much can be demonstrated about this topic in the semiclassical limit, starting from a wavepacket (coherent state) conveniently localized around a point on a closed classical orbit (section 3). It consists of considering the overlap of a time evolved quantum wavepacket with its initial state at conveniently chosen times called ‘revival times’.

In addition, a large physical literature has reported (essentially heuristic) results about the so-called ‘quantum fidelity’ (also called Loschmidt echo) in the perturbative and/or semiclassical limit (see [1–4, 7, 8, 15–17, 19–25, 30–34, 36–50]). The idea, which goes back to Peres [30], is to study the behaviour in time of the sensitivity of quantum evolutions with respect to small perturbations of the Hamiltonian, mainly in situations where the underlying classical motion is ‘chaotic’.

More precisely, one considers the overlap of the perturbed quantum evolution of some given initial wavepackets with that under the unperturbed quantum dynamics, in particular the decay properties in time of this overlap.

Here we shall consider exact results about this behaviour in the ‘linear response’ framework, starting with a bunch of initial wavepackets taken as eigenstates of the unperturbed Hamiltonian, assuming some ergodicity or mixing assumptions about the underlying classical flow, and taking the semiclassical limit $\hbar \to 0$ (section 3).

We note that these results constitute a first step towards a completely rigorous understanding of these topics in the semiclassical regime.

The paper is organized as follows. In section 2, we give a proof of a beautiful formula proposed by Mehlig–Wilkinson [28] for the metaplectic operators associated with a given symplectic map. (This topic is being developed and completed in a recent work in collaboration with Robert). This formula appears to be very useful in the semiclassical study of the quantum revivals that we develop in section 3. Section 4 contains some results about the revivals of coherent states in the case of time-periodic Hamiltonians that are *quadratic* in coordinates and momenta. In section 5 we give some result about the quantum fidelity in the semiclassical regime, in the linear-response framework, and recover a result derived heuristically by Prosen and Znidaric [34]. Section 6 contains intermediate lemmas necessary for the proof of theorem 3.6. In section 7 we give some concluding remarks.

2. A proof of the Mehlig–Wilkinson formula

In a recent paper, Mehlig and Wilkinson proposed (heuristically) a beautiful formula expressing the metaplectic operators corresponding to suitable symplectic matrices M not having 1 as an eigenvalue ([28]). In a paper in preparation with Robert [14], we give a developed and generalized proof of this formula, including the calculus of the phase, and the case where the symplectic matrix has 1 as an eigenvalue. Here we present a simple proof, closer to physical intuition, in the same spirit as in [14] but without specifying the phase factor.

A metaplectic transformation implements in the quantum world the symplectic transformations $Sp(2n)$ (canonical transformations) of classical mechanics. To a given symplectic transformation M, it corresponds unitary operators $\hat{R}(M)$ in the Hilbert space of quantum states acting covariantly on the canonical operators of position and momentum $\hat{Q}$, $\hat{P}$ in the following way:

$$\hat{R}(M)^* \begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix} \hat{R}(M) = M \begin{pmatrix} \hat{Q} \\ \hat{P} \end{pmatrix} \tag{2.1}$$

plus some algebraic conditions ensuring that $M \mapsto \hat{R}(M)$ defines a projective representation of $Sp(2n)$, with sign indetermination. For any $M \in Sp(2n)$ we can find a C^1-smooth curve F_t, $t \in [0, 1]$ in $Sp(2n)$ such that $F_0 = \mathbb{1}$ and $F_1 = M \cdot F_t$ is clearly the linear flow defined by the quadratic Hamiltonian $H_t(z) := \frac{1}{2} z \cdot S_t z$ with

$$S_t = -J \dot{F}_t F_t \tag{2.2}$$

where

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{2.3}$$

If U_t is the quantum propagator associated with $\hat{H}_t$, we define

$$\hat{R}(M) = U_1$$

which obeys equation (2.1). If two symplectic paths F_t and F_t' join 1 to M, then we have $U_1 = \pm U_1'$.

Definition 2.1. *Let $z := (q, p)$ be a generic point in the classical phase space $Z := \mathbb{R}^{2n}$. The covariant symbol of an operator $\hat{A}$ is a function $A^\sharp(z)$ defined in the whole phase space $\mathbb{R}^{2n}$ such that*

$$\hat{A} = \int \mathrm{d}z A^\sharp(z) \hat{T}(z) \tag{2.4}$$

with $\hat{T}(z)$ being the Weyl–Heisenberg operator

$$\hat{T}(z) := \exp\left(\frac{\mathrm{i}p \cdot \hat{Q} - \mathrm{i}q \cdot \hat{P}}{\hbar}\right). \tag{2.5}$$

Theorem 2.2. *Assume that M does not have 1 as an eigenvalue. Then the covariant symbol of $\hat{R}(M)$ is given by*

$$\Omega(z) = \frac{\gamma_M}{|\det(\mathbb{1} - M)|^{1/2}} \exp\left(\frac{\mathrm{i}}{2\hbar} z \cdot Az\right), \tag{2.6}$$

where γ_M is a complex number of modulus 1, and A is defined by

$$A := \frac{J}{2}(M - \mathbb{1})^{-1}(\mathbb{1} + M) \tag{2.7}$$

and J is the symplectic matrix defined in equation (2.3).

Proof. By the inversion formula we get

$$\Omega(X) = \mathrm{Tr}(U_1 \hat{T}(-X)) \tag{2.8}$$

which using the coherent states basis

$$\varphi_z := \hat{T}(z)\varphi_0 \tag{2.9}$$

where

$$\varphi_0 := (\pi\hbar)^{-n/4} \exp(-x^2/2\hbar) \tag{2.10}$$

can be rewritten as

$$\Omega(X) = (2\pi\hbar)^{-n} \int \mathrm{d}z \langle \varphi_z, \hat{T}(-X) U_1 \varphi_z \rangle. \tag{2.11}$$

In all that follows we shall omit for simplicity the factors 2π and $\hbar$, which we can easily reintroduce in the result. We have

$$\hat{T}(X)\hat{T}(z) = \mathrm{e}^{-\mathrm{i}\sigma(X,z)/2} \hat{T}(X + z) \tag{2.12}$$

with the notation

$$\sigma(z, z') = z \cdot Jz'.$$

Clearly

$$U_1\varphi_z = \hat{T}(z')\hat{R}(M)\varphi_0 \tag{2.13}$$

with $z' := Mz$, so that

$$\Omega(X) = \int \mathrm{d}z\, \mathrm{e}^{\mathrm{i}\sigma(X,z)/2 - \mathrm{i}\sigma(z',X+z)/2}\langle\varphi_Y, \hat{R}(M)\varphi_0\rangle \tag{2.14}$$

where by Y we denote the variable

$$Y := X + z - z' = X + (\mathbb{1} - M)z. \tag{2.15}$$

The scalar product $\langle\varphi_Y, \hat{R}(M)\varphi_0\rangle$ is equal to

$$\mathrm{e}^{\mathrm{i}Y\cdot QY/2}$$

with Q a matrix that we do not need to make explicit. Performing the change of variable $z \mapsto Y$, we rewrite $\sigma(z + z', X + z)$ in terms of X and Y alone. The cross-terms vanish, so that we are simply left with

$$-\sigma(z + z', X + z) = X \cdot AX + Y \cdot Q'Y \tag{2.16}$$

with Q' a matrix that we do not need to make explicit. For the calculus of A, we easily check that the quadratic terms in X are

$$-(\mathbb{1} + M)(\mathbb{1} - M)^{-1}X \cdot JM(\mathbb{1} - M)^{-1}X = -(\mathbb{1} - M)^{-1}X \cdot J(\mathbb{1} + M)(\mathbb{1} - M)^{-1}X \tag{2.17}$$

where we have used that, by the symplecticity of M,

$$(\mathbb{1} + \tilde{M})JM = JM + J = J(\mathbb{1} + M) \tag{2.18}$$

where $\tilde{M}$ denotes the transpose of M. Therefore (2.15) is nothing but

$$-(\mathbb{1} - M)^{-1}X \cdot JM(\mathbb{1} - M)^{-1}X = -(\mathbb{1} - M)^{-1}X \cdot \frac{JM - \tilde{M}J}{2}(\mathbb{1} - M)^{-1}X. \tag{2.19}$$

However, using again the symplecticity of M, we have

$$(\mathbb{1} - \tilde{M})^{-1}(JM - \tilde{M}J) = J(\mathbb{1} + M)$$

Therefore (2.19) is nothing but

$$-\tfrac{1}{2}X \cdot J(\mathbb{1} + M)(\mathbb{1} - M)^{-1}X \equiv X \cdot AX$$

Therefore inserting this result into (2.14) simply leads to

$$\Omega(X) = \Omega(0)\,\mathrm{e}^{\mathrm{i}X\cdot AX/2}, \tag{2.20}$$

where $\Omega(0)$ is the integral over Y that we do not perform exactly, although it is a pure Gaussian integral, but we shall content ourselves with determining its absolute value. □

Lemma 2.3. *We have*

$$|\Omega(0)| = |\det(\mathbb{1} - M)|^{-1/2}. \tag{2.21}$$

Proof.

$$\langle\varphi_z, U_1\varphi_z\rangle = \int \mathrm{d}X U_1(X) W_{\varphi_z}(X),$$

where $U_1(X)$ is the Weyl symbol of the quantum evolution operator U_1, and $W_{\varphi_z}(X) = W_{\varphi_0}(X-z) = \Phi_0(X-z)$ with

$$\Phi_0(z) := \mathrm{e}^{-z^2} \tag{2.22}$$

being the Wigner function of φ_0. Thus

$$\langle \varphi_z, U_1\varphi_z \rangle = U_1 * \Phi_0(z). \tag{2.23}$$

Let us now calculate $\int \mathrm{d}z |\langle \varphi_z, U_1\varphi_z\rangle|^2$ by using the Plancherel theorem

$$\int \mathrm{d}z |\langle \varphi_z, U_1\varphi_z\rangle|^2 = \int \mathrm{d}z |(U_1 * \Phi_0)(z)|^2 \tag{2.24}$$

$$= \int \mathrm{d}z |\mathcal{F}(U_1 * \Phi_0)(z)|^2,$$

where $\mathcal{F}$ denotes the symplectic Fourier transform. But $\Omega(z)$ is the symplectic Fourier transform of $U_1(z)$. Thus equation (2.24) becomes

$$= \int \mathrm{d}z |\Omega(z)|^2 |\mathcal{F}\Phi_0(z)|^2 = |\Omega(0)|^2 \int \mathrm{d}z\, \mathrm{e}^{-z^2/2} = |\Omega(0)|^2 \tag{2.25}$$

because A being real symmetric (which can be deduced easily from the symplecticity of M), $|\Omega(z)| \equiv |\Omega(0)|, \quad \forall z \in \mathbb{R}^{2n}$.

Let us now calculate the same quantity using the Wigner transforms

$$\begin{aligned} |\langle \varphi_z, U_1\varphi_z\rangle|^2 &= 2\int_{\mathbb{R}^{2n}} \mathrm{d}u\, W_{\hat{T}(z)\varphi_0}(u) W_{\hat{T}(z')\hat{R}(M)\varphi_0}(u) \\ &= 2\int \mathrm{d}u\, W_{\varphi_0}(u-z) W_{\hat{R}(M)\varphi_0}(u-z') \\ &= 2\int \mathrm{d}u\, \exp[-(u-z)^2 - (M^{-1}u - z)^2]. \end{aligned} \tag{2.26}$$

Now, in order to integrate equation (2.26) over z, we perform the change of variable

$$z \mapsto X := z - u$$

so that we are left with a simple Gaussian of the form

$$2\int \mathrm{d}X\, \mathrm{d}u\, \mathrm{e}^{-X^2-(X-Y)^2} = \int \mathrm{d}u\, \mathrm{e}^{-Y^2/2} = |\det(\mathbb{1} - M^{-1})|^{-1} \tag{2.27}$$

since $Y := (M^{-1} - \mathbb{1})u$. But since M is symplectic, it has determinant 1 and thus equation (2.27) equals

$$|\det(M - \mathbb{1})|^{-1}.$$

Hence

$$|\Omega(0)|^2 \equiv |\det(\mathbb{1} - M)|^{-1}. \tag{2.28}$$

□

Theorem 2.2 is nothing but the beautiful Mehlig–Wilkinson formula [28] for the metaplectic representation. In [14] , we establish it in full generality, including the calculus of the phase factor $\gamma(F)$. Assume that F is a symplectic map not having 1 as an eigenvalue. Then

$$\hat{R}(F) = h^{-n} \frac{\gamma(F)}{|\det(\mathbb{1} - F)|^{1/2}} \int_{\mathbb{R}^{2n}} \mathrm{d}z \exp\left(\frac{\mathrm{i}z \cdot Az}{2\hbar}\right) \hat{T}(z), \tag{2.29}$$

where $\gamma(F)$ is a complex number of modulus 1 that we do not specify, and A is the $2n \times 2n$ real symmetric matrix

$$A := \tfrac{1}{2} J(F + \mathbb{1})(F - \mathbb{1})^{-1} \tag{2.30}$$

and J is the symplectic matrix (2.3)

3. Quantum revivals

We consider a quantum wavepachet $\varphi \in \mathcal{H} = L^2(\mathbb{R}^n)$, together with a quantum Hamiltonian $\hat{H}$ selfadjoint in $\mathcal{H}$. According to the Schrödinger equation, the quantum evolution of wavepackets is provided by the unitary group of evolution $U_H(t) := \exp(-\mathrm{i}t\hat{H}/\hbar)$, $\hbar$ being the Planck constant (divided by 2π). The quantum revivals of that wavepacket are observed by considering the overlap between the evolved wavepacket $U_H(t)\varphi$ with the initial one φ:

$$R(t) := \langle \varphi, U_H(t)\varphi \rangle. \tag{3.1}$$

Assume now that the initial wavepacket is taken as a Gaussian one, namely a coherent state φ_z defined in equation (2.9)–(2.10) for any $z := (q, p) \in Z := \mathbb{R}^{2n}$.

The coherent states (sometimes written in the Dirac notation $|z\rangle$ instead of φ_z) have been shown to display remarkable semiclassical propagation properties (see [12]) that we recall here.

Along a tradition which goes back to Hepp (1974), one can start by 'following' a coherent state along its semiclassical evolution. We shall establish the following result: starting from a coherent state $|z\rangle$ at time $t = 0$, its quantum evolution stays, up to a phase, close to a 'squeezed state'

$$\hat{T}(z_t)\hat{R}(F_t)|0\rangle$$

centred around the point $z_t := \phi^t_H(z)$, (Φ^t_H being the classical flow generated by the classical symbol H of $\hat{H}$), with a 'dispersion' governed by a symplectic matrix F_t that we shall make precise later. This approximation is of order $O(\hbar^\epsilon)$ as long as time does not go beyond the so-called *Ehrenfest time* $T_E := \lambda^{-1}\log\hbar^{-1}$. Intuitively the phase-space directions where the wavepacket *spreads* are the unstable directions of the classical flow, whereas those along which they *are squeezed* are the stable ones; those *stable* and *unstable* directions are encoded in the symplectic matrix F_t.

All that follows will be true for a very general class of Hamiltonians (possibly time-dependent):

$$\exists m, M, K > 0 : (1 + z^2)^{-M/2}\left|\partial_z^\gamma H(z, t)\right| \leqslant K \qquad \forall|\gamma| \geqslant m \tag{3.2}$$

uniformly for $(t, z) \in [-T, T] \times Z$, such that the classical and quantum evolutions respectively (for the classical symbol and its Weyl quantization resp.) exist for $t \in [-T, T]$.

It is well known that the stability of the classical Hamiltonian evolution governed by $H(z, t)$ is given by the following linear system:

$$\dot{F} = JM_tF \tag{3.3}$$

where M_t is the $2n \times 2n$ Hessian matrix of H at point z_t of the classical trajectory:

$$(M_t)_{j,k} := \left(\frac{\partial^2 H}{\partial z_j \partial z_k}\right)_{j,k} (z_t, t) \tag{3.4}$$

is symmetric real, J is the symplectic matrix (2.3) and the initial datum is

$$F(0) \equiv \mathbb{1}. \tag{3.5}$$

Consider the purely quadratic Hamiltonian (time-dependent):

$$\hat{H}_0(t) := \tfrac{1}{2}\hat{Z} \cdot M_t\hat{Z}. \tag{3.6}$$

It induces a quantum evolution $U_0(t, t')$ via

$$\mathrm{i}\hbar\frac{\partial}{\partial t}U_0(t, t') = \hat{H}_0(t)U_0(t, t') \tag{3.7}$$

which is entirely explicit:

Lemma 3.1. *Let F_t be the $2n \times 2n$ symplectic matrix solution of (3.3)–(3.5). We note by $\hat{R}(F_t)$ the associated metaplectic operator, unitary in $\mathcal{H}$. We have*

$$U_0(t,0) = \hat{R}(F_t) \tag{3.8}$$

and therefore by the chain rule

$$U_0(t,t') = \hat{R}(F_t)\hat{R}(F_{t'}^{-1}). \tag{3.9}$$

The proof of this result is classical and can be found in [29].

It encompasses the physical intuition that, for Hamiltonians that are purely *quadratic*, the quantum dynamics is exactly solvable in terms of the classical one. Namely the linear equation (3.3) is nothing but the classical Hamilton's equations for the quadratic Hamiltonian (3.6).

In fact $\hat{R}(F_t)$ decomposes itself into the product of two unitaries (built from the symplectic matrix F_t)

- one expressing the 'squeezing',
- one expressing the 'rotation'.

Lemma 3.2. *We define the usual 'creation and annihilation' operators of quantum mechanics as follows:*

$$a := \frac{\hat{Q} + \mathrm{i}\hat{P}}{\sqrt{2\hbar}} \qquad a^\dagger := a^* = \frac{\hat{Q} - \mathrm{i}\hat{P}}{\sqrt{2\hbar}}. \tag{3.10}$$

From F_t can be built two $2n \times 2n$ matrices E_t and Γ_t such that

$$\hat{R}(F_t) = \hat{S}(E_t)\hat{R}(t) \tag{3.11}$$

$$\hat{S}(E_t) := \exp\left(\frac{1}{2}(a^\dagger \cdot E_t a^\dagger - a \cdot E_t^* a)\right) \tag{3.12}$$

$$\hat{R}(t) = \exp\left(\frac{\mathrm{i}}{2}(a^\dagger \cdot \tilde{\Gamma}_t a + a \cdot \Gamma_t a^\dagger)\right). \tag{3.13}$$

Morally, in dimension $n = 1$, $\hat{R}(t)$ has the simple form

$$\exp\left(\frac{\mathrm{i}\gamma_t}{2}(\hat{Q}^2 + \hat{P}^2)\right)$$

which is simply a rotation by the real angle γ_t.

Let us consider the Taylor expansion up to order 2 of Hamiltonian $H(z,t)$ around the point $z = z_t := (q_t, p_t)$ of the classical trajectory at time t:

$$H_2(t) := H(z_t,t) + (z - z_t)\cdot \nabla H(z_t,t) + \tfrac{1}{2}(z - z_t)\cdot M_t(z - z_t). \tag{3.14}$$

By quantization it yields

$$\hat{H}_2(t) = H(z_t,t)\mathbb{1} + (\hat{Z} - z_t)\cdot \nabla H(z_t,t) + \tfrac{1}{2}(\hat{Z} - z_t)\cdot M_t(\hat{Z} - z_t). \tag{3.15}$$

Let $U_2(t,s)$ be the quantum propagator for the Hamiltonian $\hat{H}_2(t)$. We have:

Proposition 3.3.

$$U_2(t,s) = \mathrm{e}^{\mathrm{i}(\delta_t - \delta_s)/\hbar}\hat{T}(z_t)\hat{R}(F_t)\hat{R}\big(F_s^{-1}\big)\hat{T}(-z_s) \tag{3.16}$$

where

$$\delta_t := S_t(z) - \frac{q_t\cdot p_t - q.p}{2} \tag{3.17}$$

and $S_t(z) = \int \mathrm{d}s(\dot{q}_s \cdot p_s - H(z_s,s))$ is the classical action along the trajectory $z \to z_t$.

The proof of this result can be found in [12].

In fact this propagator which, being constructed via generators of the coherent/squeezed states acts in a simple manner on coherent states, and appears to be a good approximation of the full propagator $U_H(t, s)$ in the classical limit, when acting on coherent states:

$$U_2(t,0) = e^{i\delta_t/\hbar}\hat{T}(z_t)\hat{S}(E_t)\hat{R}(t)\hat{T}(-z)$$
$$U_2(t,0)|z\rangle = e^{i\delta_t/\hbar}\hat{T}(z_t)\hat{S}(E_t)\hat{R}(t)|0\rangle$$
$$= e^{i\delta_t/\hbar+\gamma_t}\hat{T}(z_t)\hat{S}(E_t)|0\rangle$$

where $\gamma_t = \frac{1}{2}\operatorname{tr}\Gamma_t$.

The state

$$\Phi(z,t) := \hat{T}(z_t)\hat{S}(E_t)|0\rangle \tag{3.18}$$

is simply a squeezed state centred at z_t, with a 'squeezing' given by the matrix E_t.

Theorem 3.4. *Let H be a Hamiltonian satisfying assumptions (3.2) and the existence of classical and quantum flows for $t \in [-T, T]$. Then we have, uniformly for $(t, z) \in [-T, T] \times Z$,*

$$\|U_H(t,0)\varphi_z - e^{i\delta_t/\hbar+\gamma_t}\Phi(z,t)\| \leqslant C\mu(z,t)^P|t|\sqrt{\hbar}\theta(z,t)^3 \tag{3.19}$$

P being a constant only depending on M and m, and

$$\mu(z,t) := \operatorname{Sup}_{0\leqslant s\leqslant t}(1+|z_s|) \qquad \theta(z,t) := \operatorname{Sup}_{0\leqslant s\leqslant t}(\operatorname{tr} F_s^* F_s)^{1/2}.$$

The proof of this result is contained in [12].

Estimate (3.19) contains the dependence on $t, \hbar, z$ of the semiclassical error term. One hopes that this error remains small when $\hbar \to 0$, provided that z belongs to some compact set of phase space, and $|t|$ is not too large.

Typically

$$\theta(z,t) \simeq e^{t\lambda}$$

where λ is some Lyapunov exponent that expresses the 'classical instability' near the classical trajectory. The rhs of equation (3.19) is therefore $O(\hbar^{\epsilon/2})$ provided

$$|t| < \frac{1-\epsilon}{6\lambda}\log\hbar^{-1} \tag{3.20}$$

which is typically the Ehrenfest time, up to a factor 1/6 that is probably inessential.

Remark 3.5 (see [12])**.** Theorem (3.4) can be modified (and therefore also the state $\Phi(z, t)$) to have an estimate in

$$\mu(z,t)^{lP}\sum_{j=1}^{l}\left(\frac{|t|}{\hbar}\right)^{j}(\sqrt{\hbar}\theta(z,t))^{2j+l}$$

and therefore typically $O(\hbar^{l/2})$ with l integer as large as one wants. The squeezed state however now depends on l, and is typically a finite linear combination of wavepackets of the form

$$\hat{T}(z_t)\hat{R}(F_t)|\Psi_\mu\rangle,$$

where the Ψ_μ are excited levels of the harmonic oscillator in dimension n.

We shall now make use of the Mehlig–Wilkinson formula (2.29) in order to calculate the dominant contribution to $R(t)$, the quantum overlap at time t.

Of course this formula holds true as long as 1 is not an eigenvalue of F, but it can be generalized to this case also (see [14]).

We now rewrite equation (3.19) as

$$U_H(t,0)|z\rangle = \mathrm{e}^{\mathrm{i}\delta_t/\hbar}\hat{T}(z_t)\hat{R}(F_t)|0\rangle + \varepsilon(t,\hbar,z). \tag{3.21}$$

We shall consider the following quantity (return probability)

$$R(\alpha,t) := |\langle\varphi_\alpha, U_H(t,0)\varphi_\alpha\rangle|. \tag{3.22}$$

It can be rewritten as

$$\left|\langle\hat{T}(\alpha)\varphi_0, \hat{T}(\alpha_t)\hat{R}(F_t)\varphi_0\rangle\right| + \varepsilon(t,\alpha,\hbar). \tag{3.23}$$

Approximant (3.23) is used to obtain the following main result of this section which is a semiclassical approximation of $R(\alpha,t)$ (the return probability): let us define the following $2n\times 2n$ matrix

$$K := (-J + \mathrm{i}\mathbb{1})(\mathbb{1} - 2\mathrm{i}A)^{-1}(J + \mathrm{i}\mathbb{1}). \tag{3.24}$$

Theorem 3.6. *There exists a constant c_t, $0 < c_t \leqslant 1$ only depending upon $F(t)$ such that $R(\alpha,t)$ has the following semiclassical approximation:*

$$c_t\left|\exp\left(-\frac{1}{4\hbar}(\alpha-\alpha_t)\cdot[\mathbb{1} - (\mathbb{1}+\mathrm{i}J)(\mathbb{1}-2\mathrm{i}A)^{-1}(\mathbb{1}-\mathrm{i}J)](\alpha-\alpha_t)\right)\right|. \tag{3.25}$$

The proof is rather technical, although not difficult, relying essentially on Gaussian integrations, and deferred to section 6.

Remark 3.7. The fact that the exponential factor in (3.25) is a truly decreasing factor as $\alpha-\alpha_t$ increases has been established in [14], using the properties of the matrix A (essentially the fact that the eigenvalues of the matrix $\frac{\mathbb{1}}{2} - \mathrm{i}A$ all lie on the circle $|z-1| = 1$, with $z = 0$ excluded).

As regards the quantum revivals, we look at expression (3.25), and where it equals 1 when time t goes on. Clearly we need $\alpha_t = \alpha$, and $c_t = 1$. The first condition is realized provided α lies on a periodic orbit γ of the classical flow, and $t = T_\gamma$ is the period of this orbit. The condition $c_t = 1$ is less transparent. It obviously holds if $F_t = \mathbb{1}$, or more generally if $F_{T_\gamma} = \mathbb{1}\cos\theta + J\sin\theta$, $\quad\theta\in[0,2\pi[$, which is not automatically fulfilled along any periodic orbit, but can however occur for some $t = pT_\gamma$, p being a repetition number (integer). We thus have the following:

Theorem 3.8. *Let γ be a periodic orbit of the classical flow corresponding to the (principal) symbol of $\hat{H}$, with period T_γ, and assume that $z\in\gamma$. Assume moreover that the stability matrix F_{T_γ} takes the value*

$$F_{T_\gamma} = \mathbb{1}\cos\theta + J\sin\theta, \qquad \theta\in[0,2\pi[. \tag{3.26}$$

Then provided that $T_\gamma \leqslant \frac{1-\varepsilon}{6}\log\hbar^{-1}$, we have that the overlap $|L_1(\alpha,t)| = |\langle\varphi_\alpha, U_t\varphi_\alpha\rangle|$ taken at value T_γ obeys

$$R(\alpha,t) = 1 - O(\hbar^\varepsilon). \tag{3.27}$$

This result expresses the semiclassical 'almost recurrence' of the quantum state $U_H(T_\gamma,0)|z\rangle$ to $|z\rangle$ for not too long periods of the orbit γ on which the phase-space point z sits.

Proof. It is not hard to check that if F_{T_γ} has the form above, then the matrix N given by (6.26) equals

$$N = -\frac{\mathrm{i}J}{2}\left(\mathbb{1}(1+\mathrm{e}^{\mathrm{i}\theta}) + \mathrm{i}J(1-\mathrm{e}^{\mathrm{i}\theta})\right)$$

and therefore $|\det N| = 1$, whence $c_t \equiv |\det N_t|^{-1/2} = 1$ (see (6.30)). Note that we have no general result ensuring that it is actually the case that (3.26) holds true for any periodic orbit γ. The only result we know for sure is that it holds true with $\theta = 0$ if the classical flow is globally periodic (see [5]). However in the following section we also provide some explicit examples where the stability matrix can be shown explicitly to have the rotation form (3.26). □

Remark 3.9. Actually we have established (see [14]) that $|\det N| = 1$, (and thus $c_t = 1$) if and only if F_{T_γ} is unitary, (which is a generalization of the rotation form (3.26)).

4. Quantum revivals for some time-dependent Hamiltonians

As we have shown in section 2, the time-dependent quadratic Hamiltonians play a special role in the understanding of quantum evolutions, since in particular they generate the operators of the metaplectic group. They have the remarkable well-known property that the quantum dynamics is exactly solvable in terms of the classical dynamics. They are thus a good laboratory to investigate the behaviour of the stability matrix F_t when it is propagated along a classical (closed) orbit.

Thus in this section we investigate the return probability of a coherent state for the quantum evolution U_t generated by a quantum Hamiltonian of the following form:

$$\hat{H}(t) = \tfrac{1}{2}\hat{Z}\cdot S(t)\hat{Z}, \tag{4.1}$$

where $S(t)$ is a real symmetric $2n \times 2n$ matrix, and also particular non-quadratic Hamiltonians of the form

$$\hat{H}(t) = \frac{\hat{P}^2}{2} + f(t)\frac{\hat{Q}^2}{2} + \frac{g^2}{\hat{Q}^2}.$$

In the first case, the calculus of

$$F_\alpha(t) := |\langle\varphi_\alpha, U_t\varphi_\alpha\rangle| \tag{4.2}$$

will be exact, and we make use of the result of section 2 that the quantum propagator associated with the Hamiltonian $\hat{H}(t)$ is exactly given by the metaplectic operator $\hat{R}(F_t)$, where F_t is the stability matrix solution of equation (3.3).

We clearly have: (by some abuse of notation we still denote by $U_t(z)$ the Weyl symbol of the unitary quantum propagator U_t)

$$\langle\varphi_\alpha, U_t\varphi_\alpha\rangle = \int \mathrm{d}z\, W_{\varphi_\alpha}(z)U_t(z) = \int \mathrm{d}z\, W_{\varphi_0}(z-\alpha)U_t(z) \tag{4.3}$$

$$= \mathcal{F}^{-1}\left(W^\sharp_{\varphi_0}(z)\Omega(z)\right)(\alpha),$$

where $\mathcal{F}$ is the symplectic Fourier transform, and

$$W^\sharp_{\varphi_0}(z) = (\mathcal{F}W_{\varphi_0})(z) = \langle\varphi_0, \hat{T}(z)\varphi_0\rangle \equiv \mathrm{e}^{-z^2/4\hbar}.$$

Therefore

$$\Omega(z)W^\sharp_{\varphi_0}(z) = \frac{\gamma_F}{|\det(\mathbb{1}-F)|^{-1/2}}\exp\left(-\frac{1}{2\hbar}z\cdot\left(\frac{\mathbb{1}}{2} - \mathrm{i}A\right)z\right) \tag{4.4}$$

and thus

$$F_\alpha(t) = |\mathcal{F}^{-1}\big(W^\sharp_{\varphi_0}\Omega\big)(\alpha)| = |\det N_t|^{-1/2}\exp\left\{-\frac{1}{2\hbar}J\alpha\cdot B_t J\alpha\right\} \tag{4.5}$$

where

$$B_t := \left(\tfrac{\mathbb{1}}{2} - \mathrm{i}A\right)^{-1} = (F_t - \mathbb{1})N_t^{-1} \tag{4.6}$$

and

$$N_t := -\mathrm{i}\frac{J}{2}((\mathbb{1} - \mathrm{i}J)F_t + \mathbb{1} + \mathrm{i}J). \tag{4.7}$$

As shown in the appendix (lemma 6.2), $|\det N_t| \geqslant 1$, and $B_t = 0$ whenever $F_t = \mathbb{1}$, so that in this case $F_\alpha(t) = 1$.

Analysis of a few more or less trivial examples:

- $\hat{H}(t) = g(t)\dfrac{\hat{Q}\cdot\hat{P} + \hat{P}\cdot\hat{Q}}{2}$

  g being a T-periodic function of mean zero:

$$\int_0^T \mathrm{d}t\, g(t) = 0.$$

  Let G be the primitive of g that vanishes at $t = 0$. Then clearly the stability matrix

$$F_t = \begin{pmatrix} \mathrm{e}^{G(t)} & 0 \\ 0 & \mathrm{e}^{-G(t)} \end{pmatrix} \tag{4.8}$$

  is T-periodic and obeys $F_T = \mathbb{1}$ together with $N_T = -\mathrm{i}J$ so that

$$F_\alpha(T) = 1$$

  for *any coherent state* φ_α, namely φ_α is recurrent along the periodic orbits of the classical flow (and this is even true without any phase).

  Note that the stability matrix F_t is always a rotation in this case:

$$F_t = \begin{pmatrix} \cos G(t) & \sin G(t) \\ -\sin G(t) & \cos G(t) \end{pmatrix}$$

- $\hat{H}(t) = g(t)\dfrac{\hat{P}^2 + \hat{Q}^2}{2}$

  g being again T-periodic. Then, letting $\alpha = (q, p) \in \mathbb{R}^2$ be any point in the phase space one gets by an explicit calculus the return probability of a coherent state

$$\langle\varphi_\alpha, U_t\varphi_\alpha\rangle = \exp\left(-\frac{1}{2}\mathrm{i}G(t) - \frac{1}{2}\mathrm{i}(p^2+q^2)\sin G(t) - (p^2+q^2)\sin^2\left(\frac{G(t)}{2}\right)\right). \tag{4.9}$$

  From this we deduce that if $G(t) = \int_0^T \mathrm{d}t\, g(t) = 2k\pi$, for $k \in \mathbb{Z}$, then

$$\langle\varphi_\alpha, U_t\varphi_\alpha\rangle = (-1)^k, \quad \forall\alpha.$$

- $\hat{H}(t) = \dfrac{\hat{P}^2}{2} + \dfrac{1}{2}(\lambda\cos(\omega t) + \mu)\hat{Q}^2$

The classical trajectories are the solutions of Mathieu equations, which are known to be either *stable* or *unstable*, depending on the zones in the parameter space $(\lambda, \mu) \in \mathbb{R}^2$.

Assume (λ, μ) lie in some *stability zone* of the Mathieu equation. Then there exists a real parameter ρ together with an infinite sequence $\{c_n\}_{n\in\mathbb{Z}}$ such that the stability matrix has the following form:

$$F_t = \begin{pmatrix} \frac{1}{C}\sum_n c_n \cos[(2n+\rho)\omega t/2] & \frac{2}{\omega D}\sum_n c_n \sin[(2n+\rho)\omega t/2] \\ -\frac{\omega}{2C}\sum_n (2n+\rho)c_n \sin[(2n+\rho)\omega t/2] & \frac{1}{D}\sum_n c_n(2n+\rho)\cos[(2n+\rho)\omega t/2] \end{pmatrix} \tag{4.10}$$

where the constants C, D are determined by

$$C := \sum_n c_n$$

$$D := \sum_n (2n+\rho)c_n.$$

In general F_t is only *quasiperiodic*, but it is known that for the limiting curves $\mu = f_k(\lambda)$ of the stability zones, one has $\rho = 2k$, $k \in \mathbb{N}$, so that in this case

$$F_T = \mathbb{1}, \qquad T = \frac{2\pi}{\omega}$$

in which case we have exact revivals for any coherent wavepacket φ_α.

Note that if the $\{c_n\}_{n\in\mathbb{Z}}$ and ρ happen to satisfy

$$(2-\omega\rho)\sum_{n\in\mathbb{Z}} c_n = 2\omega\sum_{n\in\mathbb{Z}} nc_n$$

then F_T reduces to a simple rotation matrix

$$F_T = \begin{pmatrix} \cos(\pi\rho) & \sin(\pi\rho) \\ -\cos(\pi\rho) & \cos(\pi\rho) \end{pmatrix}.$$

Note that it has been established in [9] that beautiful recurrences can be obtained in this case in a much more general way if the initial wavepacket (coherent state) is chosen in an appropriate way, with a dispersion conveniently adjusted according to the classical solutions of the Mathieu equations. Define

$$a := \frac{\omega D}{2C}, \tag{4.11}$$

which is a positive real number, at least in the high frequency regime (and for parameters in the stability zones). Define

$$\psi(x) := (\pi\hbar)^{-1/4} a^{-1/2} \exp\left(-a\frac{x^2}{2\hbar}\right) \tag{4.12}$$

as the reference normalized wavepacket. It is the ground state of a particular hamonic oscillator of the form

$$\hat{H}_0 := \frac{1}{2}\hat{P}^2 + \left(\frac{D\omega}{2C}\right)\frac{1}{2}\hat{Q}^2. \tag{4.13}$$

Its recurrences are defined through the overlap

$$F(t) := \langle\psi, U_t\psi\rangle. \tag{4.14}$$

It has been established in [9] that one has the exact result:

Proposition 4.1. *One has*

$$F(t) = \mathrm{e}^{-\mathrm{i}\rho\omega t/4} G(t) \tag{4.15}$$

where G is the T-periodic function defined as

$$G(t) := \left(\sum_n c_n \left(\frac{1}{2C} + \frac{2n+\rho}{2D} \right) \mathrm{e}^{\mathrm{i}n\omega t} \right)^{-1/2}. \tag{4.16}$$

Therefore one has revivals at times $T = \frac{2\pi}{\omega}$ *up to the secular phase* $\exp\left(-\frac{\mathrm{i}\rho\omega t}{4}\right)$*:*

$$|F(kT)| = 1, \qquad \forall k \in \mathbb{Z}$$

Remark 4.2. Note that the reference wavepacket ψ is a kind of coherent (squeezed) state, localized around the fixed point $x(t) = 0$ of the Mathieu equation, which is a periodic orbit reduced to a point. In the paper referenced above [9], this point corresponds to the centre of a quadrupole radiofrequency trap (Paul trap), which 'traps' an ion at its centre, classically as well as quantum mechanically. Ideally this holds forever, but due to imperfections of the trap, and possible occurrence of several ions trapped together, there is an 'escape time' which can be evaluated with respect to the various parameters of the ion and of the trap (see [11]).

- $\hat{H}(t) = \frac{\hat{P}^2}{2} + f(t)\frac{\hat{Q}^2}{2} + \frac{g^2}{\hat{Q}^2}$

in one dimension, g being a positive constant, and $t \mapsto f$ is a T-periodic function. (This Hamiltonian has been considered in [10] as a model for interacting ions in a quadrupolar trap.) The important point is that the purely quadratic part of the Hamiltonian governs the quantum dynamics. Let $x(t)$ be a solution of Hill's equation

$$\ddot{x}(t) + f(t)x(t) = 0 \tag{4.17}$$

with the initial data

$$x(0) = \alpha, \qquad \dot{x}(0) = \frac{\mathrm{i}}{\alpha}. \tag{4.18}$$

It can be easily shown that $x(t)$ can be written as

$$x(t) := \mathrm{e}^{u+\mathrm{i}\theta} \tag{4.19}$$

where $t \mapsto u, \theta$ are real functions, and u is T-periodic, and that

$$\dot{\theta} = \mathrm{e}^{-2u}$$

provided we are in a stability region of equation (4.17) (which are generalization of the stability zones of the Mathieu equation considered above). Then it can be established (see [10]):

Proposition 4.3. *Define*

$$a := \tfrac{1}{2}\sqrt{1 + 8g^2} \tag{4.20}$$

and for any $n \in \mathbb{N}$

$$E_n = 2n + a + 1 \tag{4.21}$$

$$\varphi_n(x) = \left(\frac{2n!}{\Gamma(a+n+1)} \right)^{1/2} x^{a+1/2}\, \mathrm{e}^{-x^2/2} L_n^a(x^2) \tag{4.22}$$

L_n^a being the Laguerre polynomials. Then a solution of the time-dependent Schrödinger equation for $\hat{H}(t)$ is

$$\psi_n(x,t) = \exp\left(-\mathrm{i}\theta E_n + \frac{-u + \mathrm{i}\dot{u}x^2}{2}\right)\varphi_n(x\,\mathrm{e}^{-u}). \tag{4.23}$$

Thus, since u is T-periodic, and $\dot{u}(0) = 0$ due to our choice of initial data (4.18), this result expresses the recurrence, up to a phase, of the corresponding set of wavepackets.

Proof. It is easy to check by direct computation that the unitary evolution operator corresponding to $\hat{H}_t$ is of the form $U_t U_0^*$ with

$$U_t = \mathrm{e}^{\mathrm{i}\dot{u}\hat{Q}^2/2}\,\mathrm{e}^{\mathrm{i}u(\hat{P}\cdot\hat{Q}+\hat{Q}\cdot\hat{P})/2}\,\mathrm{e}^{-\mathrm{i}\theta\hat{H}_0}, \tag{4.24}$$

where we make use of the following differential equation satisfied by u:

$$\ddot{u} + \dot{u}^2 - \mathrm{e}^{-4u} + f = 0. \tag{4.25}$$

□

5. Quantum fidelity in the linear-response semi-classical regime

Consider the following quantum Hamiltonians $\hat{H}_0$ and $\hat{H} := \hat{H}_0 + \lambda V$, where V is some perturbation, and λ is a (small) coupling constant. We want to compare the quantum evolutions $U_{H_0}(t)$ under $\hat{H}_0$, and $U_H(t)$ under $\hat{H}$ respectively. Starting with a reference normalized state ψ, the quantum fidelity amplitude (or the Loschmidt echo), is defined as

$$F(t) := |\langle U_{H_0}(t)\psi, U_H(t)\psi\rangle|^2.$$

Of course if $t = 0$ this quantity is equal to 1, and the fidelity is assumed to decrease as time increases. The point is to obtain suitable regimes where something can be said about this decrease.

If we choose the reference state ψ to be an eigenstate of either $\hat{H}_0$ or $\hat{H}$, then clearly *the fidelity reduces to the return probability* for either $\hat{H}$, or $\hat{H}_0$. In this sense the studies of return probability and quantum fidelity are linked.

As an entertainment exercise, we study the quantity $F(t)$ when ψ is taken as an eigenstate ψ_j of the unperturbed Hamiltonian $\hat{H}_0$ corresponding to the discrete eigenvalue E_j, and recall an old (exact) estimate on the short time fidelity amplitude squared in that case:

$$F_j(t) = |\langle\psi_j, U_H(t)\psi_j\rangle|^2$$

which means that we only have to consider the 'return probability' in this case.

The result we recall here is known as the 'Mandelstam–Tamm inequality' [27] (see also [26]). Let

$$\bar{E} := \langle\psi, \hat{H}\psi\rangle \tag{5.1}$$

$$\Delta E^2 := \langle\psi, (\hat{H} - \bar{E})^2\psi\rangle. \tag{5.2}$$

Proposition 5.1. *Assume $\bar{E}, \Delta E > 0$ are defined by equation (5.1) and (5.2), respectively. Then the return probability of a state ψ with respect to the quantum evolution $U_H(t)$ generated by the Hamiltonian $\hat{H}$, $|\langle\psi, U_H(t)\psi\rangle|^2$ remains, as long as time stays bounded above by $\pi\hbar/2\Delta E$, bounded from below by* $\cos^2\left(\frac{t\Delta E}{\hbar}\right)$*:*

$$|\langle\psi, U_H(t)\psi\rangle|^2 \geqslant \cos^2\left(\frac{t\Delta E}{\hbar}\right). \tag{5.3}$$

One of the most studied regime is the so-called 'linear response' regime, which is a perturbative one, where the Dyson series of $U_H(t)$ is truncated up to second-order expansion terms. Let us denote by $V(t)$ the Heisenberg observable:

$$V(t) := U_{H_0}(-t) V U_{H_0}(t) \tag{5.4}$$

and by $\bar{V}(t)$ the 'mean' of this operator on the interval $[0, t]$:

$$\bar{V}(t) := \int_0^t V(s)\,\mathrm{d}s. \tag{5.5}$$

One defines

$$f^{\mathrm{LR}}(t) := 1 + \frac{\lambda}{\mathrm{i}\hbar}\langle\psi, \bar{V}(t)\psi\rangle - \frac{\lambda^2}{\hbar^2}\int_0^t \mathrm{d}s\langle\psi, V(s)\bar{V}(s)\psi\rangle. \tag{5.6}$$

Since we are mainly interested by the absolute value (squared) of this quantity (to obtain the 'quantum fidelity'), we now consider $F^{\mathrm{LR}}(t)$ to be the modulus squared of $f^{\mathrm{LR}}(t)$ truncated to second order in λ:

$$F^{\mathrm{LR}}(t) = 1 - \frac{\lambda^2}{\hbar^2}\left(2\mathrm{Re}\int_0^t \mathrm{d}s\langle\psi, V(s)\bar{V}(s)\psi\rangle - \langle\psi, \bar{V}(t)\psi\rangle^2\right). \tag{5.7}$$

For such a truncated linear-response fidelity, we shall perform the small Planck constant limit, namely the semiclassical regime under suitable assumptions on H_0.

We shall now take as particular reference states the eigenstates of $\hat{H}_0$ whose eigenvalues lie in some neighbourhood of a given classical energy E. Let us consider $\hbar$- dependent energy intervals in which the spectrum of $\hat{H}_0$ is a pure point:

$$I(\hbar) := [\alpha(\hbar), \beta(\hbar)]$$

where $\alpha(\hbar) < E < \beta(\hbar)$ with $\lim_{\hbar\to 0}(\beta(\hbar) - \alpha(\hbar)) = 0$, $\beta(\hbar) - \alpha(\hbar) \geqslant C\hbar$ for some $C > 0$, and denote

$$\Lambda(\hbar) := \{j : E_j \in I(\hbar)\}.$$

Let ψ_j be an eigenstate of $\hat{H}_0$ for some $j \in \Lambda(\hbar)$. For such a reference state, the expression in (5.7) equals

$$F_j^{\mathrm{LR}}(t) = 1 - \frac{\lambda^2}{\hbar^2}\left(2\mathrm{Re}\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s'\langle\psi_j, VU_{H_0}(-s')VU_{H_0}(s')\psi_j\rangle - \langle\psi_j, \bar{V}(t)\psi_j\rangle^2\right). \tag{5.8}$$

We make use of the following result:

Proposition 5.2. *Let $\hat{A}$ be the Weyl quantization of a classical symbol A, and let us consider the Wigner transform associated with a given state $\psi \in \mathcal{H}$:*

$$W_\psi(q, p) := h^{-n}\int_{\mathbb{R}^n}\mathrm{d}y\,\bar{\psi}\left(q + \frac{y}{2}\right)\psi\left(q - \frac{y}{2}\right)\mathrm{e}^{\mathrm{i}p.y/\hbar} \equiv h^{-n}\int_Z \mathrm{d}z'\langle\psi, \hat{T}(z')\psi\rangle\,\mathrm{e}^{\mathrm{i}\sigma(z,z')/\hbar}. \tag{5.9}$$

Then we have

$$\langle\psi, \hat{A}\psi\rangle = \int_Z \mathrm{d}z\, W_\psi(z)A(z). \tag{5.10}$$

Let $V(z)$ (resp. A_s) be the principal symbol of V (resp. $VV(s)$). Consider the classical flow $\Phi^t_{H_0}$ induced by the classical symbol of $\hat{H}_0$. Then we have the following semiclassical result (the Egorov theorem):

Proposition 5.3. *As* $\hbar \to 0$ *we have*

$$A_s(z) \to V(z)V \circ \Phi^s_{H_0}(z) \tag{5.11}$$

uniformly for (s,z) in any compact set of $\mathbb{R} \times Z$.

This result goes back to Egorov [18] (see also [6]).

Now the Wigner function of (almost any) ψ_j, for $j \in \Lambda(\hbar)$ can also be shown to have very interesting semiclassical limit properties, for *ergodic classical flows*, namely to converge towards the microcanonical measure on the energy surface $H_0 = E$ as $\hbar \to 0$ in a weak sense (known as the Schnirelman theorem, see [13]):

$H_0(q, p)$ being the principal symbol of $\hat{H}_0$, we denote by Σ_E the energy surface

$$\Sigma_E := \{(q, p) \in \mathbb{R}^{2n}, H_0(q, p) = E\} \tag{5.12}$$

and by $\mathrm{d}\sigma_E$ the microcanonical measure normalized to unity:

$$\mathrm{d}\sigma_E = \left(\int_{\Sigma_E} \frac{\mathrm{d}\Sigma_E}{|\nabla H_0|}\right)^{-1} \frac{\mathrm{d}\Sigma_E}{|\nabla H_0|}, \tag{5.13}$$

where $\mathrm{d}\Sigma_E$ is the Euclidean measure on Σ_E.

We make the following assumption:

(H) The dynamical system $\left(\Sigma_E, \mathrm{d}\sigma_E, \Phi^t_{H_0}\right)$ is ergodic, which means for any continuous function a on Σ_E, we have for almost every $z \in \Sigma_E$

$$\lim_{T\to\infty} T^{-1} \int_0^T a \circ \Phi^t_{H_0}(z)\,\mathrm{d}t = \int_{\Sigma_E} a(z)\,\mathrm{d}\sigma_E(z). \tag{5.14}$$

Proposition 5.4. *Under rather technical assumptions (see [13]) and hypothesis (H), for any* $\hbar > 0$, *there exists* $M(\hbar) \subseteq \Lambda(\hbar)$ *depending only on the Hamiltonian* H_0 *such that*

$$\lim_{\hbar\to 0} \left(\frac{\sharp M(\hbar)}{\sharp\Lambda(\hbar)}\right) = 1 \tag{5.15}$$

and

$$\lim_{\hbar\to 0\, j\in M(\hbar)} \langle\psi_j, \hat{A}\psi_j\rangle = \int_{\Sigma_E} A(z)\,\mathrm{d}\sigma_E(z). \tag{5.16}$$

The proof of this proposition has been given in [13].

We shall employ all these results to give the semiclassical behaviour of the linear-response quantum fidelity.

We define the mean and the autocorrelation function of V (on the energy shell) as follows:

$$\bar{V}_E := \int_{\Sigma_E} \mathrm{d}\sigma_E(z) V(z) \tag{5.17}$$

$$C_{V,E}(t) := \int_{\Sigma_E} \mathrm{d}\sigma_E(z)\left(V(z)V \circ \Phi^t_{H_0}(z) - \bar{V}_E^2\right). \tag{5.18}$$

Theorem 5.5. *Under assumption (H) we have the following result:*

$$\lim_{\hbar\to 0} \frac{\sum_{j\in M(\hbar)} F_j^{\mathrm{LR}}(t)}{\sharp\Lambda(\hbar)} = 1 - 2\frac{\lambda^2}{\hbar^2}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' C_{V,E}(s'). \tag{5.19}$$

The proof of this result follows easily from propositions 5.3–5.4 applied to equation (5.7).

Remark 5.6. This result first appeared in [34], with a more heuristic approach. Note that a useful estimate of the second term in the rhs of (5.19) could be obtained under better knowledge on the possible 'mixing properties' of the system (and therefore on the decay of correlations $C_{V,E}(t)$ as t becomes large).

Usually physicists consider only the two first terms, claiming that it can be 'exponentiated' to yield an 'exponential-type' decay of fidelity (in the linear-response semiclassical regime) of the form $\exp(-C\lambda^2 t/\hbar^2)$, provided the 'mixing' is strong enough. C is a constant given by the mixing property of the classical flow. The argument goes as follows: if the decay of the correlation function is integrable, (for example if the mixing is exponential) then the integral over s' in equation (5.19) is bounded uniformly in s, and therefore the rhs of (5.19) is bounded above by

$$1 - C\frac{\lambda^2}{\hbar^2}|t|.$$

They now 'exponentiate' this estimate to yield the 'large-time' behaviour of $F_j^{\mathrm{LR}}(t)$. We are not able to go further now in that direction by our rigorous approach.

6. Proof of theorem 3.6

We now give the proof of theorem 3.6: using repeatedly the following property of the Weyl–Heisenberg operator $\hat{T}(z)$:

Lemma 6.1.

(i)

$$\hat{T}(z)\hat{T}(z') := \mathrm{e}^{-\mathrm{i}\sigma(z,z')/2\hbar}\hat{T}(z+z')$$

(ii)

$$\langle\varphi_0, \hat{T}(z)\varphi_0\rangle = \exp\left(-\frac{z^2}{4\hbar}\right)$$

we easily get that the rhs of equation (3.23) equals (up to the error term $\varepsilon(t,\alpha,\hbar)$)

$$|\langle\varphi_0, \hat{T}(\alpha_t - \alpha)\hat{R}(F_t)\varphi_0\rangle|. \tag{6.20}$$

Now using the Mehlig–Wilkinson formula (2.25) for $\hat{R}(F_t)$, we get, provided 1 is not an eigenvalue of F_t:

$$\begin{aligned}|\langle\varphi_0, \hat{T}(\alpha_t - \alpha)\hat{R}(F_t)\varphi_0\rangle| &= |\det(\mathbb{1} - F_t)|^{-1/2}\left|\int \mathrm{d}u\, \mathrm{e}^{\mathrm{i}u\cdot Au/2\hbar}\langle\varphi_0, \hat{T}(\alpha_t - \alpha)\hat{T}(u)\varphi_0\rangle\right| \\ &= |\det(\mathbb{1} - F_t)|^{-1/2}\left|\int \mathrm{d}u\langle\varphi_0, \hat{T}(u + \alpha_t - \alpha)\varphi_0\rangle\, \mathrm{e}^{\mathrm{i}u\cdot Au/2\hbar}\, \mathrm{e}^{-\mathrm{i}\sigma(\alpha_t - \alpha, u)/2\hbar}\right|.\end{aligned} \tag{6.21}$$

We use the simplifying notation

$$z_t := \alpha_t - \alpha. \tag{6.22}$$

Now, forgetting the factors $\hbar$ that we shall reintroduce later by homogeneity, we get that equation (6.20) equals

$$\frac{1}{|\det(\mathbb{1} - F_t)|^{1/2}}\left|\int_{\mathbb{R}^n} \mathrm{d}u \exp\left(\frac{\mathrm{i}}{2}u\cdot Jz_t + \mathrm{i}\frac{u.Au}{2}\right)\langle\varphi_0, \hat{T}(u + z_t)\varphi_0\rangle\right|$$

$$= \frac{1}{|\det(\mathbb{1} - F_t)|^{1/2}} \left| \int \mathrm{d}u \exp\left(\frac{\mathrm{i}u.Jz_t}{2} - \frac{1}{4}(u + z_t)^2 + \frac{\mathrm{i}}{2} u.Au \right) \right|$$

$$= |\det(\mathbb{1} - F_t)|^{-1/2} \mathrm{e}^{-\frac{z_t^2}{4}} \left| \int \mathrm{d}u \exp\left(-\frac{1}{2} u \cdot \left(\frac{\mathbb{1}}{2} - \mathrm{i}A \right) u + \mathrm{i}u(J + \mathrm{i}\mathbb{1}) \frac{z_t}{2} \right) \right|. \tag{6.23}$$

It is easy to check that the $2n \times 2n$ matrix $\frac{\mathbb{1}}{2} - \mathrm{i}A$ obeys

$$\mathrm{Re}\left(\frac{\mathbb{1}}{2} - \mathrm{i}A \right) > 0$$

so that by the formula of the Fourier transform of Gaussian functions, this yields

$$\left| \det\left(\frac{\mathbb{1}}{2} - \mathrm{i}A \right) \right|^{-1/2} |\det(\mathbb{1} - F_t)|^{-1/2} \left| \exp\left(-\frac{z_t^2}{4} - w \cdot (\mathbb{1} - 2\mathrm{i}A)^{-1} w \right) \right| \tag{6.24}$$

where w denotes the following *complex* vector in $\mathbb{C}^{2n}$

$$w := \tfrac{1}{2}(J + \mathrm{i}\mathbb{1}) z_t.$$

Thus (6.24) can be rewritten as

$$\left| \det\left(\frac{\mathbb{1}}{2} - \mathrm{i}A \right) (\mathbb{1} - F_t) \right|^{-1/2} \left| \exp\left(-\frac{1}{4} z_t \cdot (\mathbb{1} + K) z_t \right) \right|$$

where we have defined K as in equation (3.24).

Recollecting the various prefactors, and reintroducing the parameter $\hbar$, we finally get as a semiclassical approximant of $R(\alpha, t)$ the following expression:

$$R(\alpha, t) \simeq \left| \det(\mathbb{1} - F_t) \left(\frac{\mathbb{1}}{2} - \mathrm{i}A \right) \right|^{-1/2} \left| \exp\left(-\frac{1}{4\hbar} z_t \cdot (\mathbb{1} + K) z_t \right) \right|. \tag{6.25}$$

We have the following lemma:

Lemma 6.2.
(i) We have:

$$\frac{\mathbb{1}}{2} - \mathrm{i}A = N(F - \mathbb{1})^{-1}$$

where N is a $2n \times 2n$ *matrix such that* $|\det N| \geqslant 1$. *(ii) The matrix K is such that*

$$|\exp(z.Kz)| \leqslant \exp(z^2)$$

for all $z \in \mathbb{R}^{2n}$.

Proof. $\frac{\mathbb{1}}{2} - \mathrm{i}A = -\frac{\mathrm{i}J}{2} \left(-\mathrm{i}J(F - \mathbb{1}) + F + \mathbb{1} \right) (F - \mathbb{1})^{-1} = N(F - \mathbb{1})^{-1}$ where

$$N := -\frac{\mathrm{i}J}{2} \left((\mathbb{1} - \mathrm{i}J)F + \mathbb{1} + \mathrm{i}J \right) \tag{6.26}$$

which has the four-block form

$$N = -\frac{\mathrm{i}J}{2} \begin{pmatrix} \mathbb{1} + Z & Z' + \mathrm{i}\mathbb{1} \\ \mathrm{i}(Z - \mathbb{1}) & \mathrm{i}(Z' - \mathrm{i}\mathbb{1}) \end{pmatrix}, \tag{6.27}$$

where $Z = a - \mathrm{i}c$, $Z' = b - \mathrm{i}d$ and a, b, c, d are the $n \times n$ block matrices of which F is composed:

$$F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

But using the symplecticity of F, the result that $|\det N| \geqslant 1$ can be easily checked.

(ii) is established in [14]. It uses the fact that $w.w = 0$ and that $\frac{1}{2} - \mathrm{i}A$ has all eigenvalues located on the circle $|z-1| = 1$, with the origin excluded. □

Then we conclude that the prefactor in (6.25) is $c_t := |\det N|^{-1/2} \leqslant 1$, which completes the proof of theorem 3.6.

7. Concluding remarks

Our work is a first mathematical approach of estimates for the quantum return probabilities (and quantum revivals), and of the quantum fidelity for a rather large class of Hamiltonians. It also contains a rigorous proof of the Mehlig–Wilkinson formula for the metaplectic transformations, which appear to be a useful tool, when combined with the coherent states framework, to understand the semiclassical behaviour of the return probability. Then we are able to describe under which conditions on the linearized flow we can have quantum revivals for wavefunctions being coherent states located on a *periodic orbit* of the classical flow. We stress that a similar treatment can be performed for a semiclassical analysis of the quantum fidelity. This will be developed elsewhere [14].

Here we have concentrated our attention on the linear response regime for the quantum fidelity. It has been already studied in the physics literature by several authors [8, 31, 34, 40]. Furthermore the link of the fidelity decay with the question of decoherence has been put forward [15, 17, 20, 25, 32, 50]. It is an interesting topic that deserves further investigation, but that we have not worked upon.

Since no complete review is available till now on quantum fidelity, contrary to the quantum revivals problem [35], we have provided in the bibliography, for the readers' convenience, a rather long (although not exhaustive!) list of references on the physics literature on quantum fidelity. It is not the purpose here to discuss them in detail. In my opinion, one of the prominent points in these approaches is that they interestingly distinguish several regimes (Fermi golden rule, semiclassical, linear response) where significantly different time behaviour of the fidelity manifest themselves; moreover they study the link between these behaviours and the regular/chaotic nature of the underlying classical motion. We think that these topics open new research lines in mathematical physics.

Let us stress finally that the results developed in this paper, regarding quantum revivals as well as quantum fidelity, show a delicate interplay between *time t, Planck constant, and the size of the perturbation* λ. In particular in the linear response regime, the parameter that governs the fidelity decay is the combination $t\lambda/\hbar$. Regarding the semiclassical regime, we do not seem up to now to be able to go beyond the so-called 'Ehrenfest time' $C|\log\hbar|$.

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